Mobile Robot Kinematics

We're going to start talking about our mobile robots now. There robots differ from our arms in 2 ways: They have sensors, and they can move themselves around. Because their movement is so different from the arms, we will need to talk about a new style of kinematics: Differential Drive.

1. Differential Drive is how many mobile wheeled robots locomote.
2. Differential Drive robot typically have two powered wheels, one on each side of the robot. Sometimes there are other passive wheels that keep the robot from tipping over.
3. When both wheels turn at the same speed in the same direction, the robot moves straight in that direction.
4. When one wheel turns faster than the other, the robot turns in an arc toward the slower wheel.
5. When the wheels turn in opposite directions, the robot turns in place.
6. We can formally describe the robot behavior as follows:
   (a) If the robot is moving in a curve, there is a center of that curve at that moment, known as the Instantaneous Center of Curvature (or ICC). We talk about the instantaneous center, because we'll analyze this at each instant- the curve may, and probably will, change in the next moment.
   (b) If r is the radius of the curve (measured to the middle of the robot) and l is the distance between the wheels, then the rate of rotation (ω) around the ICC is related to the velocity of the wheels by:

$$\omega(r + \frac{l}{2}) = v_r$$

$$\omega(r - \frac{l}{2}) = v_l$$

   Why? The angular velocity is defined as the positional velocity divided by the radius:

$$\frac{d\theta}{dt} = \frac{V}{r}$$

This should make some intuitive sense: the farther you are from the center of rotation, the faster you need to move to get the same angular velocity. If you travel at π radians per second for 1 second, you should travel a distance of half the circumference, or πr. Since this was in one second, the velocity was πr per second. So π radians per second equals πr velocity, so $v = \omega r$. Once we have those equations, we can solve for r or ω:

$$
\begin{aligned}
v_r &= \omega(r + \frac{l}{2}) \\
&= \omega r + \omega\frac{l}{2} \\
v_l &= \omega(r - \frac{l}{2}) \\
&= \omega r - \omega\frac{l}{2}
\end{aligned}
$$

subtract the two equations

$$
\begin{aligned}
v_r - v_l &= \frac{2\omega l}{2} \\
\omega &= \frac{v_r - v_l}{l}
\end{aligned}
$$

From the left column:

$$
\begin{aligned}
v_r &= \omega r + \omega\frac{l}{2} \\
v_l &= \omega r - \omega\frac{l}{2}
\end{aligned}
$$

add the two equations

$$
\begin{aligned}
2\omega r &= v_r + v_l \\
r &= \frac{l(v_r + v_l)}{2(v_r - v_l)}
\end{aligned}
$$

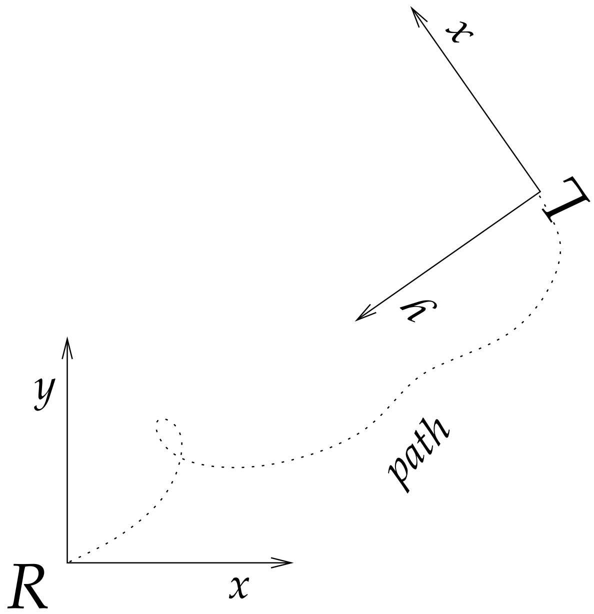
Things to note:

i. angular velocity is the difference in the wheel speeds over their distance apart.

ii. if $v_r = v_l$, then ω is 0, the robot moves straight.

iii. if $v_r = -v_l$, then r is 0 and the robot spins in place.

7. The robot is, at any one time at a location x, y, and facing a direction which forms some angle θ with the x-axis of the reference frame.Defining $\theta = 0$ to be the robot facing along the positive x-axis keeps us consistent with mathematical tradition but has an additional consequence. As the robot moves, its local frame moves with it, so θ is the angle between the reference frame x-axis and the local frame x-axis. The triple x, y, θ is called the **pose** of the robot.



(A mobile robot began at the reference frame origin and as it moved, its local frame moved with it)

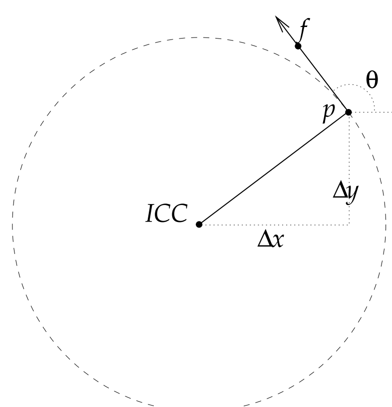
8. The forward kinematic problem is: given a robot at some pose, and moving at some angular velocity ω during a time period δt, determine the new pose for the robot.

   (a) First, note that all of these elements are functions of time: $x(t), y(t), \omega(t), V(t), \theta(t)$.

   (b) Next, lets calculate where the ICC is, given r. In the (currently non-existent) figure, the robot is facing in a direction indicated by the ray $\vec{pf}$. The ray is tangent to the curve being traversed by the robot at that moment, so the segment from p to ICC is perpendicular to $\vec{pf}$. Δx and Δy make up the right triangle, and:

$$\Delta x = -r \sin \theta$$

$$\Delta y = r \cos \theta$$



   (c) We'll represent the pose of the robot a column vector: $\begin{vmatrix} x \\ y \\ \theta \end{vmatrix}$

(d) To calculate the new location we'll perform the following steps: start at the reference frame, translate out to the original position (${}^{R}T_{P_0}$). Rotate to the current pose (position plus orientation) ${}^{P_0}T_{P_\theta}$. Translate to the ICC (${}^{P_\theta}T_{ICC_\theta}$). Rotate around the ICC (${}^{ICC_\theta}T_{ICC_\omega}$). Finally, translate back out (${}^{ICC_\omega}T_N$. This gives us the new position of the robot.

(e) This is given by the equation:

$$^{R}T_N = {}^{R}T_{P_0} \times {}^{P_0}T_{P_\theta} \times {}^{P_\theta}T_{ICC_\theta} \times {}^{ICC_\theta}T_{ICC_\omega} \times {}^{ICC_\omega}T_N.$$

$$^{R}T_N = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\omega\delta t & -\sin\omega\delta t & 0 & 0 \\ \sin\omega\delta t & \cos\omega\delta t & 0 & 0 \\ 0 & 0 & 1 & \omega\delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\omega\delta t - \sin\theta\sin\omega\delta t & -\cos\theta\sin\omega\delta t - \sin\theta\cos\omega\delta t & 0 & r\cos\theta\sin\omega\delta t + r\sin\theta\cos\omega\delta t + x - r\sin\theta \\ \sin\theta\cos\omega\delta t - \cos\theta\sin\omega\delta t & -\sin\theta\sin\omega\delta t - \sin\theta\sin\omega\delta t & 0 & r\sin\theta\sin\omega\delta t - r\cos\theta\cos\omega\delta t + y + r\cos\theta \\ 0 & 0 & 0 & \theta + \omega\delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where the rightmost column is the x, y and θ.

9. Note that v_l and v_r are really functions of time $v_l(t)$ and $v_r(t)$ (they change over time as the robot moves in different ways) thus, r and ω are both also functions of time $r(t)$ and $\omega(t)$. 9

   (a) Because we don't have nice expressions for these functions, what we normally do is break the robot's actions up in to periods of time where v_l and v_r are constant, and thus can be replaced with a single number. Once we have those single numbers, the above formula's are easy to calculate.

   (b) But, thinking about these parameters as functions gives us another way to derive these equations; a way that many feel is simpler. In this method, we note that the functions of x, y, θ depend on the functions of the velocity $V(t)$ and the angular rotation $\omega(t)$:

$$x(t) = \int_0^t V(t)\cos[\theta(t)]dt$$

$$y(t) = \int_0^t V(t)\sin[\theta(t)]dt$$

$$\theta(t) = \int_0^t \omega(t)dt.$$

   (c) In the case of a differential drive robot, $V(t) = \frac{v_r(t)+v_l(t)}{2}$, the average of the two wheels, and we know $\omega(t)$ from above. Thus with substitution, the equations become:

$$x(t) = \frac{1}{2}\int_0^t [v_r(t) + v_l(t)]\cos[\theta(t)]dt \quad (2)$$

$$y(t) = \frac{1}{2}\int_0^t [v_r(t) + v_l(t)]\sin[\theta(t)]dt \quad (3)$$

$$\theta(t) = \int_0^t \omega(t)dt. \quad (4)$$

(d) Again, we don't have descriptions of these functions, but if we assume that v_l and v_r are constant, then we can substitute in:

$$\begin{aligned}\theta(t) &= \int_0^t \frac{v_r - v_l}{l} dt. \\ &= \frac{v_r - v_l}{l} \int_0^t 1 dt. \\ &= \frac{v_r - v_l}{l} t \end{aligned} \tag{5}$$

Then we can substitute this back into the other two equations:

$$\begin{aligned} x(t) &= \frac{1}{2} \int_0^t [v_r + v_l] \cos[\frac{v_r - v_l}{l} t] dt \\ &= \frac{v_r + v_l}{2} \int_0^t \cos[\frac{v_r - v_l}{l} t] dt \\ &= \frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \sin[\frac{v_r - v_l}{l} t] \Big|_0^t \\ &= \frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \sin[\frac{v_r - v_l}{l} t] - \frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \sin[\frac{v_r - v_l}{l} 0] \\ &= \frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \sin[\frac{v_r - v_l}{l} t] \end{aligned} \tag{6}$$

and

$$\begin{aligned} y(t) &= \frac{1}{2} \int_0^t [v_r + v_l] \sin[\frac{v_r - v_l}{l}] dt \\ &= \frac{v_r + v_l}{2} \int_0^t \sin[\frac{v_r - v_l}{l}] dt \\ &= -\frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \cos[\frac{v_r - v_l}{l} t] \Big|_0^t \\ &= -\frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \cos[\frac{v_r - v_l}{l} t] - -\frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \cos[\frac{v_r - v_l}{l} 0] \\ &= -\frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \cos[\frac{v_r - v_l}{l} t] + \frac{v_r + v_l}{v_r - v_l} \frac{l}{2} \end{aligned} \tag{7}$$

(e) Why aren't these the same as the equations from (1)? They are, we just have to perform the same substitutions we did for the integral case, plus account for assumptions in the integral equations. Recall that:

$$x(t) = r\cos\theta\sin\omega\delta t + r\sin\theta\cos\omega\delta t + x - r\sin\theta$$

if v_r and v_l are constant,

$$\omega = \frac{(v_r - v_l)}{l}, \text{ so}$$

$$x(t) = r\cos\theta\sin[\frac{(v_r - v_l)}{l}\delta t] + r\sin\theta\cos[\frac{(v_r - v_l)}{l}\delta t] + x - r\sin\theta.$$

$$r = \frac{l(v_l + v_r)}{2(v_r - v_l)}, \text{ so}$$

$$x(t) = \frac{l(v_l + v_r)}{2(v_r - v_l)}\cos\theta\sin[\frac{(v_r - v_l)}{l}\delta t] + \frac{l(v_l + v_r)}{2(v_r - v_l)}\sin\theta\cos[\frac{(v_r - v_l)}{l}\delta t] + x - r\sin\theta.$$

Note that in the integral equations, we started from time 0,

so therefore, the original angle θ is 0.

$$x(t) = \frac{l(v_l + v_r)}{2(v_r - v_l)}\sin\frac{(v_r - v_l)}{l}\delta t + x.$$

and

$\delta t = t$ and $x = 0$, resulting in,

$$x(t) = \frac{l(v_l + v_r)}{2(v_r - v_l)}\sin\frac{(v_r - v_l)}{l}t.$$

which is the same as from eq (6)

10. **Important Note:** Since equations (5), (6), and (7) were derived using the assumption that the period in time under discussion began at time 0. At time 0, the robot's pose is $\langle 0, 0, 0\rangle$. Therefore those equations give us the pose of the robot *with respect to the robot's pose at the beginning of the time period.* We've seen this before, haven't we? The calculated pose is with respect to a local coordinate frame set to the robot's position at the start of the time period. We can then calculate the pose of the robot with respect to our reference frame using our old formula: ${}^{R}T_N \times {}^{N}p = {}^{R}p$.

11. In all of this analysis, it was assumed that $v_r \neq v_l$. Otherwise r would be infinite, and the equations are not valid. When these two are equal, we need to derive different equations. Starting from eqn (4), we note that $\theta(t)$ is not longer a function of time, since θ never changes. Since we're deriving these in terms of the local frame, $\theta(t) = 0$.

Plugging this into equations (2) and (3) plus $v_r = v_l = v$, we get:

$$
\begin{aligned}
x(t) &= \frac{1}{2}\int_0^t [v + v]\cos[\theta]dt \\
y(t) &= \frac{1}{2}\int_0^t [v + v]\sin[\theta]dt \\
&\text{so,} \\
x(t) &= v\delta t && (8) \\
y(t) &= 0 && (9) \\
\theta(t) &= 0 && (10)
\end{aligned}
$$

12. Putting it all together

    (a) Calculating the forward kinematics of the robot as it moves keeps track of the robot's location- something useful for the robot to know. What we need to do now is combine these equations into an algorithm we can use directly.

    (b) There are many ways to keep track of the robot's position. One Basic algorithm:

        i. The robot starts with a pose: $\langle 0, 0, 0\rangle$.

        ii. Generate the identity matrix and store as the transformation from the reference frame to the current frame: ${}^{R}T_{C}$.

        iii. Divide the time the robot is moving into time periods where the wheel velocities are constant.

        iv. For each time period:

            A. Use equations (5), (6), and (7) to calculate the new pose: $\langle x, y, \theta\rangle$.

            B. Calculate the transform from the reference frame to the new frame:

            $$ {}^{R}T_{N} = {}^{R}T_{C} \times {}^{C}T_{N} \qquad (11) $$

            This is the new pose of the robot in the reference frame. Store it as the current frame:

            $$ {}^{R}T_{C} = {}^{R}T_{N}. $$

        v. After any pass through the loop, ${}^{R}T_{C}$ describes the pose of the robot in the rightmost column.

13. Final note: Often programmers, when they treat the case of equal wheel velocities, combine equations (8) (9) (10) with equation (11) to give:

$$
\begin{aligned}
x &= v\cos(\theta)\delta t && (12) \\
y &= v\sin(\theta)\delta t && (13) \\
\theta &= \theta && (14)
\end{aligned}
$$

where $\langle x, y, \theta\rangle$ is the pose in the reference frame .

14. Inverse Kinematics. The inverse kinematics problem is, given an x,y,θ (and possibly a δt), find the v_r and v_l that gets the robot there. As we can see from equations (6) (7) (5), the problem is under constrained: there are infinitely many v_r and v_l values for any x, y and θ. Solutions for these can be found if we apply additional constraints, for instance...

    (a) if we assume that $v_r = v_l = v$, then the problem is much easier, because it is over constrained (see (8) (9) (10)). The robot can not get to certain (x, y) positions, or change its orientation.

    (b) If we assume $-v_r = v_l$, then equations (2), (3), (4) become:

    $$x(t) = 0$$
    $$y(t) = 0$$
    $$\theta(t) = \pm\frac{2v\delta t}{l}$$

    The problem is again over constrained because although the robot can achieve any orientation, it is fixed in place.

    (c) But if we combine these constraints in sequence, the robot can get anywhere by spinning in place until it is facing the right direction, then traveling in a straight line to the correct position, and finally spinning in place until the correct pose is reached. This is one simple solution to the inverse kinematics problem. We would like to not have to impose these constraints because the movement looks a little funny, but this makes the mathematics much easier.

    (d) If there are *obstacles* in the way, then this technique won't work. This is a harder problem, known as the path planning problem, and we will address it later.